Natural Strain*

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Abstract

The purpose of this paper is to present a consistent and thorough development of the strain and strain-rate measures affiliated with Hencky. Natural measures for strain and strain-rate, as I refer to them, are first expressed in terms of the fundamental body-metric tensors of Lodge. These strain and strain-rate measures are mixed tensor fields. They are mapped from the body to space in both the Eulerian and Lagrangian configurations, and then transformed from general to Cartesian fields. There they are compared with the various strain and strain-rate measures found in the literature. A simple Cartesian description for Hencky strain-rate in the Lagrangian state is obtained.

1 Introduction

Logarithmic strain is the preferred measure of strain used by materials scientists, who typically refer to it as the 'true strain'. It was Nadai (1937) who gave it the name 'natural strain', which seems more appropriate. This strain measure was proposed by Ludwik (1909, pg. 17) for the one-dimensional extension of a rod with length ℓ. It was defined via the integral $\int_{\ell_0}^{\ell} d\mathfrak{l}/\mathfrak{l}$, to which Ludwik gave the name "effective specific strain". Today it is also named after Hencky (1928), who extended Ludwik's measure to three-dimensional analysis by defining logarithmic strains appropriate for the three principal directions. Murnaghan (1941) later derived Hencky's strain measure as a consequence arising from the conservation of energy.

Truesdell and Toupin (1960, pg. 269) point out that Hencky's logarithmic strain measure had, up to that point in time, not been applied without difficulties because of its complexity in evaluation. With computers now being readily available, such a consideration—valid in 1960—is no longer a constraint. Historically, its use has been limited primarily to studies wherein the principal axes of strain do not rotate in the body; for example, Nadai (1937) and Davis (1937) used it to compare tensile and compressive stress/strain curves, while Hencky (1931) and Murnaghan (1941) applied it to the

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high-pressure experiments of Bridgman.

In their treatise, Truesdell and Tupin (1960, pg. 270) went on to say that: "Such simplicity for certain problems as may result from a particular strain measure is bought at the cost of complexity for other problems. In a Euclidean space, distances are measured by a quadratic form, and attempt to elude this fact is unlikely to succeed." They advocate using the quadratic strain fields of Almansi (1911) or Green (1841) instead of the logarithmic strain field of Hencky (1928). This author bases his definition for natural strain on the Riemannian, body-metric, tensor field. There is no 'eluding' this fact. The outcome is an intuitive measure for strain.

This paper presents a new and convenient method to handle strains, building on the works of Hencky and Lodge.

2 Coordinates

Space $\mathbb{S}$ is the infinite set of fixed point *places* $\mathfrak{X}$ in which we live, and is taken to be a primitive concept. A *body* $\mathbb{B}$ is a set of point *particles* $\mathfrak{P}$, another primitive notion, which occupy a bounded and connected region in space at any given instant in time. Body $\mathbb{B}$ is distinct from space $\mathbb{S}$.

A *body coordinate system* $\mathcal{B}$ is a one-to-one correspondence between the 'particles' $\mathfrak{P}$ in body $\mathbb{B}$ and their *body coordinates* ξ, which are ordered sets of three real numbers such that

$$\mathcal{B}: \mathfrak{P} \to \{\xi^1 \quad \xi^2 \quad \xi^3\}^\mathsf{T}. \tag{1}$$

Material lines and material surfaces are one- and two-dimensional subsets, respectively, of the three-dimensional body $\mathbb{B}$. Similarly, a *space coordinate system* $\mathbb{S}$ is a one-to-one correspondence between the 'places' $\mathfrak{X}$ in space $\mathbb{S}$ and their *spatial coordinates* $\mathbf{x}$ which, like the ξ, are ordered sets of three real numbers, viz.,

$$\mathcal{S}: \mathfrak{X} \to \{x^1 \quad x^2 \quad x^3\}^\mathsf{T}. \tag{2}$$

Coordinates are purely a numerical labeling for both particles and places, depending on whether one is addressing the body or space, and as such, they are independent of time.

In contrast, describing the location of a particle in space (which is at the very foundation of continuum mechanics) requires time dependence. In particular, let us consider particle $\mathfrak{P}$ in body $\mathbb{B}$ to occupy place $\mathfrak{X}$ in space $\mathbb{S}$ at the current time t, and to occupy place $\mathfrak{X}_0$ in space $\mathbb{S}$ at some reference time t_0 (such that $t_0 < t$), and therefore, we may write[1]

$$\left.\begin{array}{l} \mathbb{S}: \mathfrak{X} \to \{x^1 \quad x^2 \quad x^3\}^\mathsf{T} \\ \mathbb{S}: \mathfrak{X}_0 \to \{X^1 \quad X^2 \quad X^3\}^\mathsf{T} \end{array}\right\}. \tag{3}$$

This dependence of spatial position on time in order to describe the motion of a particle through space is a hindrance when developing constitutive models; nevertheless, most boundary value problems are solved more simply in space $\mathbb{S}$ than they are in body $\mathbb{B}$. Note that *the spatial coordinates themselves do not depend on time; rather, it is the description of a particle's motion through space that requires time dependence.* Particle $\mathfrak{P}$ always has coordinates $\boldsymbol{\xi}$, i.e., a body coordinate system $\mathfrak{B}$ is embedded in the body $\mathbb{B}$, and therefore 'convects' with any deformation—a notion introduced by Hencky (1925). There is no required reference to space $\mathbb{S}$ in this description. In contrast, a space coordinate system $\mathbb{S}$ is fixed in space $\mathbb{S}$, through which the body $\mathbb{B}$ moves.

A *configuration* (or *state*) is a one-to-one correspondence between the particles of a body and the places in space that they occupy at some instant. Let the body coordinate system $\mathfrak{B}: \mathfrak{P} \to \boldsymbol{\xi}$ be congruent with the space coordinate system $\mathbb{S}: \mathfrak{X} \to \mathbf{x}$ in the *Eulerian configuration* $\mathbb{C}: \{\mathfrak{P}\} \stackrel{t}{\equiv} \{\mathfrak{X}\}$ at current time t. The body coordinates, $\boldsymbol{\xi}[\mathfrak{B}] = \{\xi^1 \quad \xi^2 \quad \xi^3\}^\mathsf{T}$, and the current spatial coordinates, $\mathbf{x}[\mathbb{S}] = \{x^1 \quad x^2 \quad x^3\}^\mathsf{T}$, are therefore taken to be equal in the current configuration $\mathbb{C}$; thus, one writes the mapping

$$x^i = \mathfrak{F}^i(\boldsymbol{\xi}, t) \quad i = 1, 2, 3. \tag{4}$$

Consider the same spatial coordinate system $\mathbb{S}$, such that now $\mathbb{S}: \mathfrak{X}_0 \to \mathbf{X}$ in what is known as the *Lagrangian configuration* $\mathbb{C}_0: \{\mathfrak{P}\} \stackrel{t_0}{\equiv} \{\mathfrak{X}_0\}$ at reference time t_0. The body coordinates, $\boldsymbol{\xi}[\mathfrak{B}] = \{\xi^1 \quad \xi^2 \quad \xi^3\}^\mathsf{T}$, and the reference spatial coordinates, $\mathbf{X}[\mathbb{S}] = \{X^1 \quad X^2 \quad X^3\}^\mathsf{T}$, are taken to be equal in the initial configuration $\mathbb{C}_0$, leading to

$$X^i = \mathfrak{F}^i(\boldsymbol{\xi}, t_0) \quad i = 1, 2, 3. \tag{5}$$

Function $\mathfrak{F}^i$, consequently, maps the body coordinates of a particle into a one-parameter family of spatial coordinates defined over an interval in time.

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[1]In many classical texts on the subject of continuum mechanics, like that of Truesdell and Toupin (1960), $\mathbf{x}$ is used to denote a particle's spatial coordinates at current time t, while $\mathbf{X}$ is used to denote its spatial coordinates at some reference time t_0. This methodology of using lower- and upper-case letters to distinguish between the two states, or configurations, is extended to their definitions of tensor fields. This is a convenient approach to notation; it is the approach that I have adopted.

The *motion* $\mathbb{M}(\mathfrak{P}, t_0 \to t)$ of particle $\mathfrak{P}$ through space $\mathbb{S}$ is described by a one-parameter family of configurations taken over the interval in time that begins at t_0 and ends at t. This is to be distinguished from the *deformation* $\mathbb{D}(\mathfrak{P}, t_0, t)$ at particle $\mathfrak{P}$, which is described entirely by the two end states at times t_0 and t; it being independent of any and all intermediate states. Strain measures in space are descriptions of motion. Strain measures in the body are descriptions of deformation.

Continuous motions are considered in the sense that neighboring particles in the body $\mathbb{B}$ are assumed to occupy neighboring places in space $\mathbb{S}$; consequently,

$$\left.\begin{array}{l} dx^i = \dfrac{\partial \mathfrak{F}^i(\boldsymbol{\xi}, t)}{\partial \xi^j} \, d\xi^j \\[2mm] dX^i = \dfrac{\partial \mathfrak{F}^i(\boldsymbol{\xi}, t_0)}{\partial \xi^j} \, d\xi^j \end{array}\right\} \tag{6}$$

where the $\mathfrak{F}$'s are taken to be continuous and differentiable functions of the body coordinates and of time.

3 Metrics

Oldroyd (1950) and Lodge (1951) consider body $\mathbb{B}$ to be a Riemannian manifold with metric structure described by

$$(ds)^2 = d\xi^i \gamma_{ij}(\boldsymbol{\xi}, t) d\xi^j \quad , \quad |\boldsymbol{\gamma}| > 0 \tag{7}$$

where $\boldsymbol{\gamma}(\mathfrak{P}, t)$ is the absolute, symmetric, positive-definite, *covariant, body-metric tensor* at particle $\mathfrak{P}$ and time t; $d\boldsymbol{\xi}(\mathfrak{P})$ is an absolute contravariant vector at particle $\mathfrak{P}$, independent of time, representing the 'coordinate differences' between a pair $(\mathfrak{P}, \overline{\mathfrak{P}})$ of neighboring particles; and $ds(\mathfrak{P}, t)$ is a positive absolute scalar at particle $\mathfrak{P}$ denoting the 'separation' or distance between the pair $(\mathfrak{P}, \overline{\mathfrak{P}})$ of neighboring particles at time t. The determinant of the metric, i.e., $|\boldsymbol{\gamma}|$, is a positive scalar with weight two.

A body is said to be *deforming* whenever and wherever $\boldsymbol{\gamma}$ varies in time; whereas, it is *rigid* whenever and wherever $\boldsymbol{\gamma}$ is constant throughout time.

Because the body metric $\boldsymbol{\gamma}(\mathfrak{P}, t)[\mathfrak{B}] = \gamma_{ij}(\boldsymbol{\xi}, t)$ is positive definite, its inverse $\boldsymbol{\gamma}^{-1}(\mathfrak{P}, t)[\mathfrak{B}] = \gamma^{ij}(\boldsymbol{\xi}, t)$ exists and is positive definite, too; therefore, $\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{\gamma} = \boldsymbol{\delta}$ where $\boldsymbol{\delta}[\mathfrak{B}] = \delta^i_j(\boldsymbol{\xi})$ is the mixed 'idem' or identity tensor, whose components have the value of Kronecker's delta. As a consequence of this property, a dual metric structure can be prescribed (Lodge 1964, pg. 318); in particular,

$$\left(\frac{dc}{dh}\right)^2 = \sigma_{|i} \gamma^{ij}(\boldsymbol{\xi}, t) \sigma_{|j} \tag{8}$$

where

$$dc = \sigma_{|i}(\boldsymbol{\xi}) d\xi^i \quad \text{and} \quad \sigma_{|i} = \frac{\partial \sigma(\boldsymbol{\xi})}{\partial \xi^i} \, .$$

The value of $dc(\mathfrak{P})$ is constant throughout time. The absolute covariant vector $\nabla\sigma(\mathfrak{P})[\mathcal{B}] = \sigma_{|i}(\xi)$ signifies a gradient normal to the material surface σ at particle $\mathfrak{P}$, which is also independent of time. All time dependence on the right-hand side resides in the 'dual metric' $\gamma^{-1}(\mathfrak{P},t)$, which is the absolute, symmetric, positive-definite, *contravariant, body-metric tensor.*

Equation 8 was acquired by taking $\sigma = c$ and $\overline{\sigma} = c + dc$ to be two members of a one-parameter family of material surfaces, each containing their respective constituent in the pair $(\mathfrak{P}, \overline{\mathfrak{P}})$ of neighboring particles whose coordinates are given by the pair $(\xi, \xi + d\xi)$. The 'height' separating these two material surfaces, i.e., $dh(\mathfrak{P},t)$, is the minimum separation $ds_{\min}(\mathfrak{P},t)$ between particles $\mathfrak{P}$ and $\overline{\mathfrak{P}}$ at time t, as obtained by varying particle $\overline{\mathfrak{P}}$ along surface $\overline{\sigma}$. *The inverse metric, therefore, represents a reciprocal measure of the height between, or distance separating, two neighboring material surfaces.*

In space $\mathbb{S}$ where particle $\mathfrak{P}$ of body $\mathbb{B}$ is said to occupy place $\mathfrak{X}$ at the current time t, and place $\mathfrak{X}_0$ at some reference time t_0, the separation between 'neighboring particles' in the respective Eulerian $\mathbb{C}$ and Lagrangian $\mathbb{C}_0$ configurations is described in turn by

$$\left.\begin{aligned}(ds)^2 &= dx^i g_{ij} dx^j \quad, \quad |\mathbf{g}| > 0\\(ds_0)^2 &= dX^i G_{ij} dX^j \quad, \quad |\mathbf{G}| > 0\end{aligned}\right\} \quad (9)$$

where ds_0 denotes $ds(\mathfrak{X}_0, t_0)$. Here the absolute, contravariant, vector fields $dx = dx(\mathfrak{X},t)$ and $dX = dX(\mathfrak{X}_0, t_0)$ represent the coordinate differences between the same two 'neighboring particles' when mapped to space in the Eulerian and Lagrangian viewpoints, respectively. The *spatial metric* $\mathbf{g}(\mathfrak{X})$ is an absolute, symmetric, positive-definite, covariant, tensor field, which is also independent of time, with $\mathbf{G}(\mathfrak{X}_0)$ signifying its Lagrangian description.

Two distinguishing features between the body formalism developed by Lodge (1951, 1964, 1972, 1974) and the spatial formalism used by virtually all other continuum mechanicians are: *i) the body metric tensor $\gamma(\mathfrak{P},t)$ varies with time, in general, whereas the space metric tensor $\mathbf{g}(\mathfrak{X})$ does not,* and consequently, *ii) the coordinate differences for any pair $(\mathfrak{P}, \overline{\mathfrak{P}})$ of neighboring particles do not vary with time in the body, i.e., $d\xi(\mathfrak{P})$, however they will, in general, vary with time in space, i.e., $dx(\mathfrak{X},t)$.* There is a subtle yet significant feature contained within this second statement that has far reaching impact: *The separation ds is between 'neighboring particles' in both body $\mathbb{B}$ and space $\mathbb{S}$, as far as geometry is applied in the mechanics of deformable continua.*

4 Strain

Hencky (1931) spoke pertaining to the validity of conventional strain measures and said: "If we had never

heard of the theory of elasticity and if all substances surrounding us had the elasticity of soft rubber so that we could obtain finite deformations with very small forces, we could define strain as either the ratio of the change of length to the original length[2] or as the ratio of the change of length to the length after equilibrium is attained[3]. Such an ambiguity warns us that we must revise our fundamental notions. This is easy in the case in question, if we define the measure of an infinitesimal strain as the ratio of the increase in length to the length itself." This author has adopted Hencky's perspective in his definition of a mixed strain field.

Instead of defining strain as the difference between two quadratic forms, thereby describing a change in the geometry of body $\mathbb{B}$ as is classically done (see the footnotes below), this researcher chose to acquire it through an integration of a prescribed measure for strain-rate (Freed 1985, pp. 19–20). The particular, one-state, strain-rate field in question is given by the tensor contraction[4]

$$\dot{\eta}^i_j(\xi,t) \overset{\text{def}}{=} \tfrac{1}{2}\gamma^{ik}\dot{\gamma}_{kj} \equiv -\tfrac{1}{2}\dot{\gamma}^{ik}\gamma_{kj}. \quad (10)$$

Here $\dot{\eta}(\mathfrak{P},t)$ is the absolute, mixed, *natural, body, strain-rate tensor* at particle $\mathfrak{P}$ and time t.

Upon integrating this mixed strain-rate field over an interval in time, one obtains the two-state field

$$\begin{aligned}\eta^i_j(\xi,t_0,t) &= \int_{t_0}^t \dot{\eta}^i_j(\xi,t)\,dt\\&\equiv \frac{1}{2}\int_{\gamma_{\alpha\beta}(\xi,t_0)}^{\gamma_{\alpha\beta}(\xi,t)} \gamma^{ik}(\xi,t)\,d\gamma_{kj}(\xi,t)\\&= \tfrac{1}{2}\ln(\gamma_0^{ik}\gamma_{kj})\end{aligned} \quad (11)$$

with $\eta(\mathfrak{P},t_0,t)$ being the absolute, mixed, *natural, body-strain tensor* resulting from deformation $\mathbb{D}(\mathfrak{P},t_0,t)$.

The mixed body-strain field η (like the unmixed body-strain fields ε and ζ) is an invariant measure of deformation that has the desirable property of being *additive and anti-symmetric in its time arguments*, regardless of the magnitude of deformation; in particular,

$$\eta(\mathfrak{P},t_0,t) = \eta(\mathfrak{P},t_0,\mathfrak{t}) + \eta(\mathfrak{P},\mathfrak{t},t) \quad (12)$$

for all intermediate states $\mathfrak{t}$ such that $t_0 \leq \mathfrak{t} \leq t$. For finite deformations, this property is unique to the strain

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[2]Like the *covariant* body-strain ε defined by

$$\frac{(ds)^2 - (ds_0)^2}{2(ds_0)^2} = \frac{d\xi^i}{ds_0}\varepsilon_{ij}\frac{d\xi^j}{ds_0}$$

with $\varepsilon_{ij}(\xi,t_0,t) = \tfrac{1}{2}(\gamma_{ij} - \gamma^0_{ij})$ where $\gamma(\mathfrak{P},t_0)[\mathcal{B}] = \gamma^0_{ij}$.

[3]Like the *contravariant* body-strain ζ defined by

$$\frac{(dh)^2 - (dh_0)^2}{2(dh)^2} = \frac{\sigma_{|i}}{dc/dh_0}\zeta^{ij}\frac{\sigma_{|j}}{dc/dh_0}$$

with $\zeta^{ij}(\xi,t_0,t) = \tfrac{1}{2}(\gamma_0^{ij} - \gamma^{ij})$ where $\gamma^{-1}(\mathfrak{P},t_0)[\mathcal{B}] = \gamma_0^{ij}$.

[4]By convention (McConnell 1957, pp. 10-11), the superscript of a mixed field is the row index while the subscript is the column index in its matrix representation.

fields of body $\mathbb{B}$. Fitzgerald (1980) has shown that this does not hold true for the spatial Hencky strain of the Eulerian configuration $\mathbb{C}$. Hencky (1928), himself, had addressed this issue briefly.

Natural strain is also a *relative measure of deformation* in that

$$\eta(\mathfrak{P}, t_0, t_0) = \mathbf{0}. \tag{13}$$

This implies that the reference state at time t_0 is strain-free.

It is because the properties of Eqns. 12 & 13 are satisfied that strain measures of the body $\mathbb{B}$ are descriptions of deformation $\mathbb{D}(\mathfrak{P}, t_0, t)$, i.e., path independent. And, it is because strain measures in space $\mathbb{S}$ are not additive and anti-symmetric in their time arguments that they are descriptions of motion $\mathbb{M}(\mathfrak{P}, t_0 \rightarrow t)$, viz., path dependent. It is worth pointing out that the stretch tensor μ, defined below, does not possess either of the above two properties that ought to accompany a strain measure.

4.1 Stretch

The stretching of body $\mathbb{B}$ at particle $\mathfrak{P}$ over an interval $[t_0, t]$ in time is defined by the tensor contraction

$$\mu_j^i \overset{\text{def}}{=} (\gamma_0^{ik} \gamma_{kj})^{1/2} \tag{14}$$

or equivalently,

$$(\mu_j^i)^2 = \mu_k^i \mu_j^k = \gamma_0^{ik} \gamma_{kj}.$$

Here $\mu(\mathfrak{P}, t_0, t)$ is the absolute, mixed, *body-stretch tensor* describing deformation $\mathbb{D}(\mathfrak{P}, t_0, t)$.

Combining the above definition for stretch with Eqn. 11 results in the following expression for natural strain.

$$\eta_j^i = \ln(\mu_j^i) \tag{15}$$

Note that the stretch of body $\mathbb{B}$ is not defined via a polar decomposition, which is the standard means for its definition in the Cartesian mechanics of space $\mathbb{S}$. This is because only in Cartesian analysis can a tensor, say $\mathbf{R}$, be defined with the special matrix property that $\mathbf{R}^{-1} = \mathbf{R}^{\mathsf{T}}$.

4.2 Simple Extension

In the one-dimensional extension of a rod with initial length ℓ_0 and final length ℓ, the principle eigenvalue for the stretch tensor μ is $\lambda = \ell/\ell_0$. How this stretch gets interpreted as a measure of strain is not unique. In Table 1, both strains and strain-rates are listed for the natural (mixed), covariant and contravariant body-strains, along with the classical engineering strain measure (linear: small strain), as they apply to the deformation of simple extension. What distinguishes these body-strain fields is their tensorial character.

Measure	Strain	Strain Rate
Natural (mixed)	$\ln(\lambda)$	$\dot{\lambda}/\lambda$
Covariant	$\frac{1}{2}(\lambda^2 - 1)$	$\lambda\dot{\lambda}$
Contravariant	$\frac{1}{2}(1 - 1/\lambda^2)$	$\dot{\lambda}/\lambda^3$
Engineering	$(\lambda - 1)$	$\dot{\lambda}$

Table 1: Comparing various body-strain measures against engineering strain in simple extension.
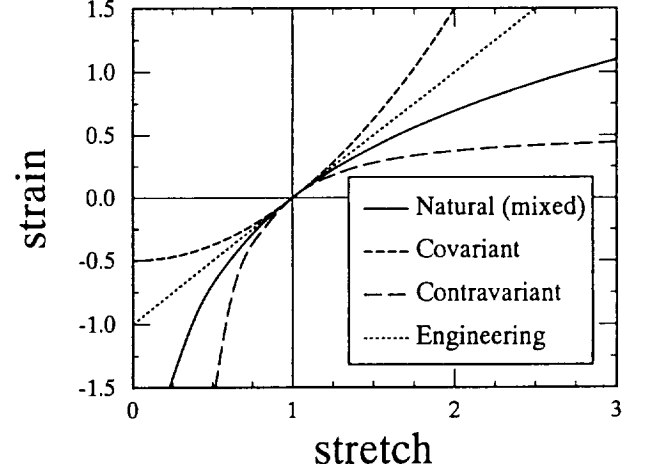

Figure 1: Comparing various body-strain measures against engineering strain in simple extension.

A graphical representation of these four strain measures is presented in Fig. 1.[5] Only the mixed strain η has the *natural*, or intuitive, property of logarithmic strain—strain that becomes infinite as stretch approaches infinity, or that becomes negative infinite as stretch goes to zero. The contravariant strain ζ asymptotes to one-half in tension, while the covariant strain ϵ asymptotes to negative one-half in compression. All are approximately equal to the engineering strain when the deformation is small.

5 Field Transfer

Even though body and space tensor fields belong to disjoint vector spaces, Lodge (1951, 1964, 1972, 1974) has demonstrated that each body tensor $\alpha(\mathfrak{P}, t)$ is isomorphic with a unique, general, space tensor $a(\mathcal{X}, t, t)$ of the same kind in the current configuration $\mathbb{C}$, where particle $\mathfrak{P}$ occupies place $\mathcal{X}$ at time t and where $t_0 \leq t \leq t$. Hence, there exists a one-to-one correspondence, or *isomorphism* $\mathbb{T}(t)$ (in Lodge's 1974 notation), that maps the body tensor $\alpha(\mathfrak{P}, t)$ into a unique, general, space tensor $a(\mathcal{X}, t, t)$ at current time t. We express this map-

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[5]Skrzypek (1993, pg. 29) presents a like figure, where these same curves are related to various spatial-strain measures.

ping as

$$\mathbf{a}(\mathfrak{X},\mathfrak{t},t) = \mathbb{T}(t)\,\boldsymbol{\alpha}(\mathfrak{P},t) \qquad (16)$$

when in the Eulerian viewpoint $\mathbb{C}\colon \{\mathfrak{P}\} \overset{t}{\equiv} \{\mathfrak{X}\}$, and whose components are related by

$$a^{u\ldots}_{\ell\ldots}(\mathbf{x},\mathfrak{t},t) = \mathbb{T}^{u\ldots m\ldots}_{\ell\ldots v\ldots}(t)\,\alpha^{v\ldots}_{m\ldots}(\boldsymbol{\xi},t)$$

given $\mathcal{B} \overset{t}{\equiv} \mathbb{S}$, where the isomorphism is defined by the field transfer operator[6]

$$\mathbb{T}^{u\ldots m\ldots}_{\ell\ldots v\ldots}(t) = \left|\frac{\partial \mathfrak{F}(\boldsymbol{\xi},t)}{\partial \boldsymbol{\xi}}\right|^{-w}_{\mathfrak{P}\equiv\mathfrak{X}}$$
$$\times \prod\left(\frac{\partial \mathfrak{F}^{\ell}(\boldsymbol{\xi},t)}{\partial \xi^m}\right)^{-1}_{\mathfrak{P}\equiv\mathfrak{X}} \prod{}'\left(\frac{\partial \mathfrak{F}^{u}(\boldsymbol{\xi},t)}{\partial \xi^v}\right)_{\mathfrak{P}\equiv\mathfrak{X}}$$

wherein w is the tensorial weight of $\boldsymbol{\alpha}$. This is not a tensor transformation, even though it looks like one, because it is defined over two manifolds, i.e., body $\mathbb{B}$ and space $\mathbb{S}$. This correspondence produces general space tensors in the *Eulerian* perspective, which is usually preferred by fluid mechanicians, where the reference configuration $\mathbb{C}_0$ has little or no intrinsic significance as the undeformed shape of the body is oftentimes unknown.

Likewise, there exists an isomorphism $\mathbb{T}(t_0)$ that maps the body tensor $\boldsymbol{\alpha}(\mathfrak{P},t)$ into a unique, general, space tensor $\mathbf{A}(\mathfrak{X}_0,t_0,t)$ of the same kind in the reference configuration $\mathbb{C}_0$ at some time t_0 with $t_0 \leq t \leq t$, which is expressed as

$$\mathbf{A}(\mathfrak{X}_0,t_0,t) = \mathbb{T}(t_0)\,\boldsymbol{\alpha}(\mathfrak{P},t) \qquad (17)$$

when in the Lagrangian viewpoint $\mathbb{C}_0\colon \{\mathfrak{P}\} \overset{t_0}{\equiv} \{\mathfrak{X}_0\}$, and whose components are related by

$$A^{u\ldots}_{\ell\ldots}(\mathbf{X},t_0,t) = \mathbb{T}^{u\ldots m\ldots}_{\ell\ldots v\ldots}(t_0)\,\alpha^{v\ldots}_{m\ldots}(\boldsymbol{\xi},t)$$

given $\mathcal{B} \overset{t_0}{\equiv} \mathbb{S}$, where this isomorphism is defined by the field transfer operator

$$\mathbb{T}^{u\ldots m\ldots}_{\ell\ldots v\ldots}(t_0) = \left|\frac{\partial \mathfrak{F}(\boldsymbol{\xi},t_0)}{\partial \boldsymbol{\xi}}\right|^{-w}_{\mathfrak{P}\equiv\mathfrak{X}_0}$$
$$\times \prod\left(\frac{\partial \mathfrak{F}^{\ell}(\boldsymbol{\xi},t_0)}{\partial \xi^m}\right)^{-1}_{\mathfrak{P}\equiv\mathfrak{X}_0}\prod{}'\left(\frac{\partial \mathfrak{F}^{u}(\boldsymbol{\xi},t_0)}{\partial \xi^v}\right)_{\mathfrak{P}\equiv\mathfrak{X}_0}$$

which, as before, is not a tensor transformation because it is defined over both body $\mathbb{B}$ and space $\mathbb{S}$. This correspondence produces general space tensors in the *Lagrangian* perspective, which is usually preferred by solid mechanicians, where the reference configuration $\mathbb{C}_0$ has conceptual meaning as the undeformed shape of the body is usually known in advance.

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[6]The product $\prod$ ($\prod'$) denotes contraction with the covariant (contravariant) indices of $\boldsymbol{\alpha}$.

An important property of the isomorphisms $\mathbb{T}(t)$ and $\mathbb{T}(t_0)$ between the tensor fields of body $\mathbb{B}$ and space $\mathbb{S}$ is that *they reproduce invariant relations of all types, whether they involve addition, subtraction, contraction, covariant differentiation, scalar multiplication, or the formation of products of tensor fields.* It is precisely this reproductive property that justifies the use of the term 'isomorphism' for the one-to-one correspondence between body fields and general space fields (Lodge 1964, pg. 315).

5.1 Cartesian Fields

It is important to note that whenever body tensors are mapped into Cartesian spatial tensors, the field transfer is not necessarily one-to-one, and therefore, it need not be isomorphic; rather, it is a many (body) to one (Cartesian) mapping. This is a consequence of there being no distinction between covariance and contravariance in Cartesian tensors, and also because of a restricted set of admissible weights that Cartesian tensors can take on. In other words, an *isomorphic transfer of field is guaranteed to exist between body tensors and general space tensors, but not between body tensors and Cartesian space tensors.*

To transform a body tensor to a Cartesian space tensor, one must first map isomorphically from body $\mathbb{B}$ to general space $\mathbb{S}$, and then convert the resulting general space tensor into an admissible Cartesian space tensor. In mathematical notation, this author expresses this procedure in the Eulerian configuration $\mathbb{C}\colon \{\mathfrak{P}\} \overset{t}{\equiv} \{\mathfrak{X}\}$ as[7]

$$\boldsymbol{\alpha}(\mathfrak{P},t) \overset{t}{\leftrightarrow} \mathbf{a}(\mathfrak{X},\mathfrak{t},t) \Rightarrow \mathbf{a}(\mathfrak{X},\mathfrak{t},t) \qquad (18)$$

where

$$\boldsymbol{\alpha}(\mathfrak{P},t) \overset{t}{\leftrightarrow} \mathbf{a}(\mathfrak{X},\mathfrak{t},t)$$

implies

$$\mathbf{a}(\mathfrak{X},\mathfrak{t},t) = \mathbb{T}(t)\,\boldsymbol{\alpha}(\mathfrak{P},t).$$

In the Lagrangian configuration $\mathbb{C}_0\colon \{\mathfrak{P}\} \overset{t_0}{\equiv} \{\mathfrak{X}_0\}$, this transfer of field is signified by

$$\boldsymbol{\alpha}(\mathfrak{P},t) \overset{t_0}{\leftrightarrow} \mathbf{A}(\mathfrak{X}_0,t_0,t) \Rightarrow \mathbf{A}(\mathfrak{X}_0,t_0,t) \qquad (19)$$

where

$$\boldsymbol{\alpha}(\mathfrak{P},t) \overset{t_0}{\leftrightarrow} \mathbf{A}(\mathfrak{X}_0,t_0,t)$$

implies

$$\mathbf{A}(\mathfrak{X}_0,t_0,t) = \mathbb{T}(t_0)\,\boldsymbol{\alpha}(\mathfrak{P},t).$$

The shorthand notations $\overset{t}{\leftrightarrow}$ and $\overset{t_0}{\leftrightarrow}$ prove useful.

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[7]Cartesian spatial tensors are represented in the Sans serif font, e.g., **A**. General space tensors are represented in the Roman font, e.g., A. Upper-case letters denote a Lagrangian description, while lower-case letters denote an Eulerian description. Body tensors are represented in the Greek font, e.g., $\boldsymbol{\alpha}$.

6 Transferred Fields

The machinery is now in place to be able to relate the body fields developed in this paper with known spatial fields, at least in most cases.

6.1 Coordinates

The infinitesimal vector $d\xi$, which denotes coordinate differences between neighboring particles, maps from body $\mathbb{B}$ to space $\mathbb{S}$ according to[8]

$$\left.\begin{array}{l} d\xi^i \overset{t}{\hookleftarrow} dx^i \Rightarrow dx_i \\[4pt] d\xi^i \overset{t_0}{\hookleftarrow} dX^i \Rightarrow dX_i \end{array}\right\} \qquad (20)$$

because of Eqns. 6, 16 & 17.

6.2 Metrics

From the relationships presented thus far, the fundamental metric tensors of body $\mathbb{B}$, i.e., γ and γ^{-1}, are quickly verified to map to space $\mathbb{S}$ as

$$\left.\begin{array}{l} \gamma_{ij}(\boldsymbol{\xi},t) \overset{t}{\hookleftarrow} g_{ij}(\mathbf{x}) \Rightarrow I_{ij} \\[4pt] \gamma^{ij}(\boldsymbol{\xi},t) \overset{t}{\hookleftarrow} g^{ij}(\mathbf{x}) \Rightarrow I_{ij} \\[4pt] \gamma_{ij}(\boldsymbol{\xi},t_0) \overset{t_0}{\hookleftarrow} G_{ij}(\mathbf{X}) \Rightarrow I_{ij} \\[4pt] \gamma^{ij}(\boldsymbol{\xi},t_0) \overset{t_0}{\hookleftarrow} G^{ij}(\mathbf{X}) \Rightarrow I_{ij} \end{array}\right\} \qquad (21)$$

where the metric of Cartesian space, i.e., $\mathbf{I}[\mathcal{C}] = I_{ij}$ ($\mathcal{C} \subset \mathbb{S}$), is the identity matrix. The above relationships provide an excellent example of the many-to-one mapping property that exists between body and Cartesian fields. Notice that in each of the above four relationships, the time-stamp of the body metric corresponds with the time of field transfer, and as a consequence, no time dependence is present in the metric tensors of space.

When the time dependence of the body metric differs from the time at which field transfer takes place, then via the field transfer operators of Eqns. 16 & 17, one obtains the well-known, symmetric, two-state, deformation fields of Cauchy (1827, pp. 60–69) and Finger

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(1894), specifically (cf. Lodge 1964, pp. 319–321)

$$\left.\begin{array}{rl} \gamma_{ij}(\boldsymbol{\xi},t_0) \overset{t}{\hookleftarrow} c_{ij}(\mathbf{x},t_0,t) &= \dfrac{\partial X^k}{\partial x^i} G_{k\ell} \dfrac{\partial X^\ell}{\partial x^j} \\[8pt] \Rightarrow c_{ij}(\mathbf{x},t_0,t) &= \dfrac{\partial X_k}{\partial x_i} \dfrac{\partial X_k}{\partial x_j} \\[8pt] \gamma^{ij}(\boldsymbol{\xi},t_0) \overset{t}{\hookleftarrow} b^{ij}(\mathbf{x},t_0,t) &= \dfrac{\partial x^i}{\partial X^k} G^{k\ell} \dfrac{\partial x^j}{\partial X^\ell} \\[8pt] \Rightarrow b_{ij}(\mathbf{x},t_0,t) &= \dfrac{\partial x_i}{\partial X_k} \dfrac{\partial x_j}{\partial X_k} \\[8pt] \gamma_{ij}(\boldsymbol{\xi},t) \overset{t_0}{\hookleftarrow} C_{ij}(\mathbf{X},t_0,t) &= \dfrac{\partial x^k}{\partial X^i} g_{k\ell} \dfrac{\partial x^\ell}{\partial X^j} \\[8pt] \Rightarrow C_{ij}(\mathbf{X},t_0,t) &= \dfrac{\partial x_k}{\partial X_i} \dfrac{\partial x_k}{\partial X_j} \\[8pt] \gamma^{ij}(\boldsymbol{\xi},t) \overset{t_0}{\hookleftarrow} B^{ij}(\mathbf{X},t_0,t) &= \dfrac{\partial X^i}{\partial x^k} g^{k\ell} \dfrac{\partial X^j}{\partial x^\ell} \\[8pt] \Rightarrow B_{ij}(\mathbf{X},t_0,t) &= \dfrac{\partial X_i}{\partial x_k} \dfrac{\partial X_j}{\partial x_k} \end{array}\right\} \quad (22)$$

These fields have the property that $b^{ik}c_{kj} = I^i_j(\mathbf{x})$ and $B^{ik}C_{kj} = I^i_j(\mathbf{X})$ in general space, and that $b_{ik}c_{kj} = B_{ik}C_{kj} = I_{ij}$ in Cartesian space. The fields $\mathbf{c}$, $\mathbf{c}$ and $\mathbf{C}$, $\mathbf{C}$ are called the Eulerian and Lagrangian Cauchy deformation tensors, while the $\mathbf{b}$, $\mathbf{b}$ and $\mathbf{B}$, $\mathbf{B}$ are the Eulerian and Lagrangian Finger deformation tensors, respectively. Pertaining to these field transformations, notice that one instance of the spatial state dependence arises from the body metric tensor, while the other comes from the field transfer operator.

6.2.1 Deformation Gradients

In the tensor analysis of deformable continua, the preferred Cartesian fields for describing a body's shape are not symmetric metric-like tensors; rather, they are the non-symmetric deformation-gradient tensors which, like the deformation tensors, are two-state fields.

In the current configuration $\mathbb{C}$, the Eulerian deformation-gradient tensor $\mathbf{f}(\mathfrak{X},t_0,t)$ is defined by the transformation mapping

$$d\mathbf{X} = \mathbf{f} \cdot d\mathbf{x} \qquad (23)$$

where

$$\mathbf{f}[\mathcal{C}] = f_{ij} = \frac{\partial X_i}{\partial x_j} \quad , \quad |\mathbf{f}| > 0.$$

Similarly, in the reference configuration $\mathbb{C}_0$, the Lagrangian deformation-gradient tensor $\mathbf{F}(\mathfrak{X}_0,t_0,t)$ is defined by the transformation mapping

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} \qquad (24)$$

where

$$\mathbf{F}[\mathcal{C}] = F_{ij} = \frac{\partial x_i}{\partial X_j} \quad , \quad |\mathbf{F}| > 0$$

from which it follows immediately that $\mathbf{F} = \mathbf{f}^{-1}$. These are the definitions (but not the notations) used by

Malvern (1969, pg. 156), which he claims originate in the eighteenth century writings of Euler. Most authors, unfortunately, do not distinguish between $\mathbf{f}$ and $\mathbf{F}$, and when they refer to the deformation-gradient tensor, it is, more often than not, the Lagrangian deformation-gradient that they are using.

The component expressions for the Cauchy and Finger deformation fields given in Eqn. 22 can be written as

$$\left.\begin{aligned}
\mathbf{c}(\mathfrak{X}, t_0, t) &= \mathbf{f}^{\mathsf{T}} \cdot \mathbf{f} \\
\mathbf{b}(\mathfrak{X}, t_0, t) &= \mathbf{f}^{-1} \cdot \mathbf{f}^{-\mathsf{T}} \\
\mathbf{C}(\mathfrak{X}_0, t_0, t) &= \mathbf{F}^{\mathsf{T}} \cdot \mathbf{F} \\
\mathbf{B}(\mathfrak{X}_0, t_0, t) &= \mathbf{F}^{-1} \cdot \mathbf{F}^{-\mathsf{T}}
\end{aligned}\right\} . \tag{25}$$

Notice the symmetry in these definitions. Even though the Eulerian and Lagrangian deformation-gradient tensors are related to one-another, it is useful to distinguish between these fields, as Murnaghan (1941) and others have done, so that one does not mistakenly mix fields from both configurations when constructing a constitutive equation, for example.

6.3 Stretch

Using the results of Eqns. 21 & 22, the stretch tensor $\mu(\mathfrak{P}, t_0, t)$ of body $\mathbb{B}$ presented in Eqn. 14 transfers to space $\mathbb{S}$ as follows.

$$\left.\begin{aligned}
\mu_j^i(\boldsymbol{\xi}, t_0, t) &\overset{t}{\leftrightarrow} v_j^i(\mathbf{x}, t_0, t) = (b^{ik} g_{kj})^{1/2} \\
&\Rightarrow v_{ij}(\mathbf{x}, t_0, t) = (b_{ij})^{1/2} \\
\mu_j^i(\boldsymbol{\xi}, t_0, t) &\overset{t_0}{\leftrightarrow} U_j^i(\mathbf{X}, t_0, t) = (G^{ik} C_{kj})^{1/2} \\
&\Rightarrow U_{ij}(\mathbf{X}, t_0, t) = (C_{ij})^{1/2}
\end{aligned}\right\} \tag{26}$$

The Cartesian stretch tensors $\mathbf{v}$ and $\mathbf{U}$ are the left-Eulerian and right-Lagrangian stretch tensors, respectively, named after the polar decompositions: $\mathbf{f} = \mathbf{r} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{r}$ and $\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$, where $\mathbf{r}^{\mathsf{T}} = \mathbf{r}^{-1}$ and $\mathbf{R}^{\mathsf{T}} = \mathbf{R}^{-1}$ are orthogonal rotation tensors. As a consequence, $\mathbf{u} = \mathbf{v}^{-1}$ and $\mathbf{V} = \mathbf{U}^{-1}$. The Cartesian stretch tensors, i.e., $\mathbf{u}$, $\mathbf{v}$, $\mathbf{U}$ and $\mathbf{V}$, are symmetric and positive definite.

Since μ_j^i, v_j^i and U_j^i are mixed fields, they cannot possess the property of symmetry because of the tensor transformation law. (A mixed tensor and its transpose belong to different vector spaces.) Despite the fact that polar decomposition is a powerful theorem of matrix theory, it can play no role in general tensor analysis.

6.4 Strain

The natural strain field of body $\mathbb{B}$ transfers to space $\mathbb{S}$ as the Hencky (1928) measures of strain[9], viz.,

$$\left.\begin{aligned}
\eta_j^i(\boldsymbol{\xi}, t_0, t) &\overset{t}{\leftrightarrow} h_j^i(\mathbf{x}, t_0, t) = \ln(v_j^i) \\
&\Rightarrow h_{ij}(\mathbf{x}, t_0, t) = \ln(v_{ij}) \\
\eta_j^i(\boldsymbol{\xi}, t_0, t) &\overset{t_0}{\leftrightarrow} H_j^i(\mathbf{X}, t_0, t) = \ln(U_j^i) \\
&\Rightarrow H_{ij}(\mathbf{X}, t_0, t) = \ln(U_{ij})
\end{aligned}\right\} . \tag{27}$$

These results follow trivially once the transformations for stretch are known. The Eulerian, Hencky, strain relation of Cartesian space, i.e., $\mathbf{h} = \ln(\mathbf{v})$, appears to have been derived first by Murnaghan (1941), and whose theoretical underpinnings have been made precise by Fitzgerald (1980). Its Lagrangian counterpart, i.e., $\mathbf{H} = \ln(\mathbf{U})$, has been used by Hill (1970) and several others since then. Both of these strain measures were presented, but not applied, in the treatise of Truesdell and Toupin (1960, pg. 269). In fact, they discourage their use—an opinion not shared by this author.

6.5 Strain Rate

In preparation for mapping the strain-rate $\dot{\eta}$ of body $\mathbb{B}$ to space $\mathbb{S}$, the metric-rate $\dot{\gamma}$ is transferred first.

$$\left.\begin{aligned}
\dot{\gamma}_{ij}(\boldsymbol{\xi}, t) &\overset{t}{\leftrightarrow} \overset{\circ}{g}_{ij}(\mathbf{x}, t) = g_{ik} v_{\cdot|j}^k + v_{\cdot|i}^k g_{kj} \\
&\Rightarrow 2d_{ij}(\mathbf{x}, t) = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \\
\dot{\gamma}_{ij}(\boldsymbol{\xi}, t) &\overset{t_0}{\leftrightarrow} \dot{C}_{ij}(\mathbf{X}, t_0, t) = \\
&\quad \frac{\partial v^k}{\partial X^i} g_{k\ell} \frac{\partial x^\ell}{\partial X^j} + \frac{\partial x^k}{\partial X^i} g_{k\ell} \frac{\partial v^\ell}{\partial X^j} \\
&\Rightarrow \dot{C}_{ij}(\mathbf{X}, t_0, t) = \\
&\quad \frac{\partial v_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} + \frac{\partial x_k}{\partial X_i} \frac{\partial v_k}{\partial X_j}
\end{aligned}\right\} \tag{28}$$

The particle's velocity is given by the contravariant vector field

$$v^i(\mathbf{x}, t) = \frac{\partial \mathfrak{F}^i(\boldsymbol{\xi}, t)}{\partial t} = \frac{\partial x^i}{\partial t}$$

which is not to be confused with the left-stretch tensor v_j^i. Its covariant (spatial) derivative is denoted as

$$v_{\cdot|j}^i(\mathbf{x}, t) = \frac{\partial v^i}{\partial x^j} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} v^k$$

where $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ are the Christoffel symbols of the second kind at place $\mathfrak{X}$ in space $\mathbb{S}$ (cf. McConnell, pp. 140–146).

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[9]It is a straight forward process to prove to oneself that the covariant body-strain tensor maps to space in the Eulerian configuration C as the strain tensor of Almansi (1911), while it maps to space in the Lagrangian configuration C_0 as the strain tensor of Green (1841). Similarly, the contravariant body-strain tensor maps to space in the Eulerian configuration as the strain tensor of Signorini (1930). However, in its Lagrangian mapping, it transforms to a spatial tensor with no known origin.

The term $\overset{\circ}{\mathbf{g}}(\mathcal{X},t)$ in Eqn. 28 is Oldroyd's (1950) convected derivative of the metric $\mathbf{g}(\mathcal{X})$, which is a proper Lie derivative. Its Cartesian equivalent, viz., $\mathbf{d}(\mathcal{X},t)$, is the symmetric tensor known as the rate-of-deformation tensor.

With this groundwork in place, the natural strain-rate tensor of the body transfers to space as

$$\left. \begin{aligned} \dot{\eta}^i_j(\boldsymbol{\xi},t) &\overset{t}{\leftrightarrow} \overset{\circ}{h}{}^i_j(\mathbf{x},t) &= \tfrac{1}{2}g^{ik}\overset{\circ}{g}_{kj} \\ &\Rightarrow \overset{\circ}{h}_{ij}(\mathbf{x},t) &= d_{ij} \\ \dot{\eta}^i_j(\boldsymbol{\xi},t) &\overset{t_0}{\leftrightarrow} \dot{H}^i_j(\mathbf{X},t_0,t) = \tfrac{1}{2}B^{ik}\dot{C}_{kj} \\ &\Rightarrow \dot{H}_{ij}(\mathbf{X},t_0,t) = \frac{\partial X_i}{\partial x_k}d_{k\ell}\frac{\partial x_\ell}{\partial X_j} \end{aligned} \right\} \quad (29)$$

In the Eulerian configuration $\mathbb{C}$ of Cartesian space, the Hencky strain-rate is identical to both the Green strain-rate and the Signorini strain-rate—all being equivalent to the rate-of-deformation tensor. This is another example of the many-to-one mapping between the body and Cartesian space. Even so, these three strain-rates are unique in general space.

7 Discussion

When expressed in terms of the deformation gradient and rate-of-deformation tensors, the Eulerian, Hencky, strain and strain-rate tensors of Cartesian space are given by

$$\left. \begin{aligned} \mathbf{h}(\mathcal{X},t_0,t) &= -\tfrac{1}{2}\ln\!\left(\mathbf{f}^{\mathsf{T}}\!\cdot\mathbf{f}\right) \\ \overset{\circ}{\mathbf{h}}(\mathcal{X},t) &= \mathbf{d} \end{aligned} \right\} \quad (30)$$

while the Lagrangian, Hencky, strain and strain-rate tensors of Cartesian space are given by

$$\left. \begin{aligned} \mathbf{H}(\mathcal{X}_0,t_0,t) &= \tfrac{1}{2}\ln\!\left(\mathbf{F}^{\mathsf{T}}\!\cdot\mathbf{F}\right) \\ \dot{\mathbf{H}}(\mathcal{X}_0,t_0,t) &= \mathbf{F}^{-1}\!\cdot\mathbf{d}\cdot\mathbf{F} \end{aligned} \right\} . \quad (31)$$

Of these results, $\dot{\mathbf{H}} = \mathbf{F}^{-1}\cdot\mathbf{d}\cdot\mathbf{F}$ is a new one. The pre- and post-conditions of $\mathbf{F}^{-1}$ and $\mathbf{F}$ applied to $\mathbf{d}$ map the rate-of-deformation tensor back to its associated Lagrangian strain-rate, viz., $\dot{\mathbf{H}}$. This particular pair of pre- and post-multipliers are the pull-back conditions that arise from the transfer of a mixed body field.[10]

The Hencky strains and strain-rates of general space are all new results, as are the body fields from which they were derived.

Known attempts to find a correlation between $\dot{\mathbf{H}}$ and $\mathbf{d}$ include the works of Hill (1970), who in a footnote

stated: "In the past the use even of the tensor logarithm has been thought to involve intractable analytic difficulties."—a viewpoint that Hill attempted to change. Later on, Stören and Rice (1975) concluded that given $\mathbf{H} = \ln(\mathbf{U})$, then *their* "general relation between $\dot{\mathbf{H}}$ and $\mathbf{d}$ is very complicated. This implies that the expressions for the conjugate stress and its time-rate also become very complicated, and thus makes the measure $\mathbf{H}$ essentially intractable as a general measure in deformation-theory formulations." Gurtin and Spear (1983) found the Jaumann derivative[11] of $\mathbf{H}$ "to be an excellent approximation to $\mathbf{d}$ when $\mathbf{H}$ and $\dot{\mathbf{H}}$ are small," but not for finite deformations. In contrast, this paper derives a relationship between $\dot{\mathbf{H}}$ and $\mathbf{d}$ that is no more complicated than that which exists between the time rate-of-change of the Green strain and the rate-of-deformation tensor.

In contrast to this pessimism, it is the author's hope that the Lagrangian Hencky strain and its rate will finally find a home in continuum mechanics.

8 Summary

A natural measure for strain has been derived in the body manifold $\mathbb{B}$ of Lodge, and then mapped to the spatial manifold $\mathbb{S}$ as both general and Cartesian fields, the latter being the manifold of choice for the vast majority of continuum mechanicians. Having done so, the author agrees with the conclusions of Lodge that it is far simpler to construct physical tensor fields in the body $\mathbb{B}$ than it is to do so in space $\mathbb{S}$. Nevertheless, most practitioners work in Cartesian space; hence, the strain and strain-rate fields developed in the body $\mathbb{B}$ were mapped to space $\mathbb{S}$ with several new results being reported. In particular, the Hencky strains and strain-rates of general analysis have been defined in both the Eulerian $\mathbb{C}$ and Lagrangian $\mathbb{C}_0$ configurations, and a simple relationship has been derived for the Lagrangian Hencky strain-rate of Cartesian space.

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[10] Pre- and post-multipliers of $\mathbf{F}^{\mathsf{T}}$ and $\mathbf{F}$, respectively, are the pull-back conditions that would arise from the transfer of a covariant body field. Similarly, pre- and post-multipliers of $\mathbf{F}^{-1}$ and $\mathbf{F}^{-\mathsf{T}}$, respectively, are the pull-back conditions that would arise from the transfer of a contravariant body field.

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[11] The Jaumann derivative $\overset{\vartriangle}{\mathbf{a}}$ of tensor $\mathbf{a}$ exists as a proper Lie derivative only when (cf. Lodge 1964, pg. 330)

$$\tfrac{1}{2}\left(\dot{a}_{ij} + \gamma_{ik}\dot{a}^{k\ell}\gamma_{\ell j}\right) \overset{t}{\leftrightarrow} \overset{\vartriangle}{a}_{ij} \Rightarrow \overset{\vartriangle}{a}_{ij}$$

given that $\alpha_{ij} \overset{t}{\leftrightarrow} a_{ij} \Rightarrow a_{ij}$ and that $\alpha^{ij} \overset{t}{\leftrightarrow} a^{ij} \Rightarrow a_{ij}$. In other words, the Jaumann derivative is a proper Lie derivative in Cartesian space for only those body tensors whose covariant and contravariant forms (assuming they exist) both transfer to Cartesian space as the same field, which is a very stringent requirement.

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